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SpaceX rocket data satisfies elementary Hohmann transfer formula

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Abstract

The private company SpaceX regularly launches satellites into geostationary orbits. SpaceX posts videos of these flights with telemetry data displaying the time from launch, altitude, and rocket speed in real time. In this paper this telemetry information is used to determine the velocity boost of the rocket as it leaves its circular parking orbit around the Earth to enter a Hohmann transfer orbit, an elliptical orbit on which the spacecraft reaches a high altitude. A simple derivation is given for the Hohmann transfer velocity boost that introductory students can derive on their own with a little teacher guidance. They can then use the SpaceX telemetry data to verify the theoretical results, finding the discrepancy between observation and theory to be 3% or less. The students will love the rocket videos as the launches and transfer burns are very exciting to watch.

Introduction

SpaceX is a company that 'designs, manufactures and launches advanced rockets and spacecraft. The company was founded in 2002 to revolutionize space technology, with the ultimate goal of enabling people to live on other planets' [1]. Its founder Elon Musk is also known for his role as cofounder of PayPal and Tesla, the company that developed the Tesla electric car. Musk, entrepreneur extraordinaire, after two years at Queen's University in Ontario, Canada transferred on a scholarship to the University of Pennsylvania (Penn) [2]. He earned an undergraduate degree in economics from The Wharton School, the prestigious school of management at Penn [2]. He then earned another bachelor's degree at Penn – one in physics [2].

Figure 1. Launch of SpaceX's F9 rocket carrying the Bangabandhu satellite for deployment in a geostationary orbit, 11 May 2018. Reproduced with permission from [5]. See the photographer's website [5] and the SpaceX website [6] for many beautiful rocket photos.

See figure 1 for a photo by Ben Cooper of SpaceX's F9 rising into space from Launch Complex 39A at the NASA Kennedy Space Center in Cape Canaveral, Florida. This launch complex dates back to the 1960s when it was used for the Saturn V rocket of the Apollo program, which first sent astronauts to the Moon in 1969. The mission depicted in figure 1 is the deployment of Bangabandhu Satellite-1, ‘Bangladesh’s first geostationary communications satellite’ [3]. The F9 is the ninth version in SpaceX’s Falcon rocket series. Student fans of *Star Wars* films by George Lucas will be delighted to know that the Falcon series is named after the legendary *Millennium Falcon* ‘commanded by Harrison Ford’s character Han Solo’ [4]. The reader can find many wonderful photos of SpaceX missions posted at the website of photographer Ben Cooper [5] and the SpaceX flickr account [6].

We propose in this paper a real-life rocket application for introductory students as early as a pre-university physics course. Beginning students learn about circular motion, Newton’s Second Law, Newton’s Universal Law of Gravitation, kinetic energy, and potential energy. Using these basic concepts without needing calculus we outline a lesson plan where students can derive the necessary theoretical formulas with a little guidance from the teacher. Then, students can access rocket data from telemetry in videos to verify the theoretical results.

A simplified derivation is given for the change in velocity needed to propel a rocket from a circular parking orbit into an elliptical, Hohmann transfer orbit. Then, telemetry data provided by SpaceX in real time for many rocket trajectories are used to compare observational results to those calculated from the formula. The telemetry data appear in a corner section of each SpaceX video of the rocket launches. The videos are exciting in themselves as rocket flights into space are shows to behold.

Hohmann transfer orbit

A mission to place a satellite into orbit can be broken down into four parts. First, a rocket is launched to enter a temporary circular parking orbit around the Earth at low altitude. Second, an engine burn occurs to increase the velocity so that the rocket transfers to an elliptical orbit that will take the spacecraft to a higher altitude. See figure 2 for an illustration of the parking orbit, the transfer orbit, and the destination orbit. The third part of the mission is the release of the satellite from the main rocket. The fourth and final maneuver is another burn so that the satellite is ultimately placed in a geostationary orbit, one where the satellite orbits above the same position on the equator.

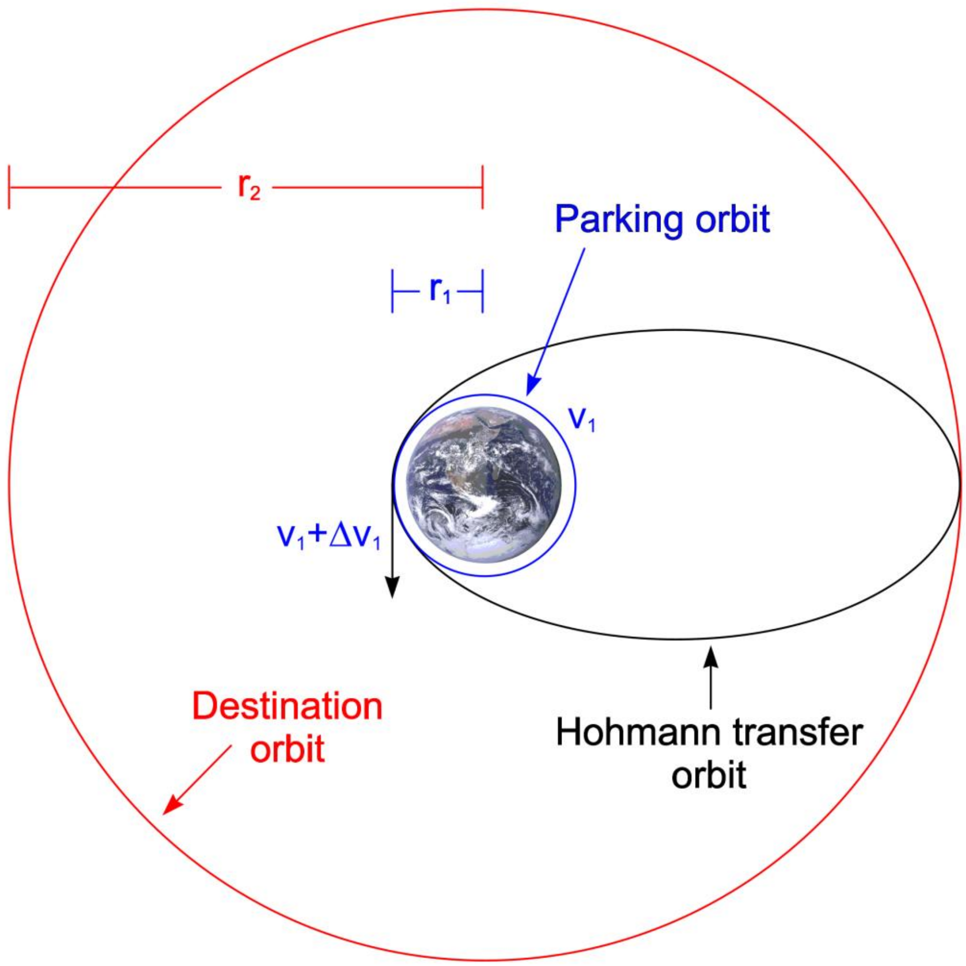

Figure 2. Parking orbit, Hohmann transfer orbit, and destination orbit. Earth image is Public Domain Courtesy NASA. Actual view of Earth would be looking down at the North Pole for a geostationary destination orbit.

The elliptical transfer orbit in figure 2 is named after the engineer Walter Hohmann who wrote his classic paper on transfer orbits in 1925 [7]. The destination orbit is intentionally not labeled as a geostationary orbit in figure 2 since SpaceX often chooses different intermediate maximum altitudes for their various flights, i.e. different r_2 values, before eventually maneuvering into a geostationary orbit. The parking orbit distance r_1 also depends on the specific flight. When the value of r_2 is close to the geosynchronous distance, measured from the center of the Earth, SpaceX labels the flight with the designation GTO for geosynchronous transfer orbit. When r_2 values are significantly less than the geosynchronous distance the label is GTO– (GTO minus); for r_2 values significantly greater, the transfer orbit is labeled GTO+ (GTO plus). Note that a geosynchronous orbit has a period equal to the Earth's rotation, while the term geostationary refers to a geosynchronous orbit over the equator such that the satellite remains stationary with respect to a position on the Earth's equator below.

The destination orbit altitudes above the Earth's surface can be found online [8] in space launch reports. Knowing the destination altitude is important in the theoretical analysis below. In this paper, the focus is on the boost speed needed to enter the Hohmann transfer orbit that takes the satellite to the destination altitude. Details in the subsequent trajectory maneuvers to finally arrive at the geostationary altitude 36 000 km above the Earth's surface are not provided by SpaceX and these adjustments occur long after the typically one-hour long SpaceX videos.

The mathematical formulas

Introductory students learn that the total energy for a satellite (mass m) in a circular orbit around the Earth (mass M) at a distance r from the center of the Earth is given by

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \qquad (1)$$

They also learn Newton's second law $F_c = ma_c$, where the centripetal force is $F_c = \frac{GMm}{r^2}$ and the centripetal acceleration[1] is $a_c = \frac{v^2}{r}$. The mass of the Earth M is taken to be very large compared to the mass of the satellite m. From Newton's second law with centripetal acceleration the speed of the satellite can be expressed as

$$v = \sqrt{\frac{GM}{r}}. \tag{2}$$

At this point it is instructive for the student to use equation (2) in conjunction with equation (1) to arrive at the equivalent form

$$E = -\frac{GMm}{2r} \tag{3}$$

for the total energy. Equation (3) is a well-known result for circular orbits. The subtle step now is to find the generalization of equation (3) for an elliptical orbit, which is needed for the Hohmann theoretical analysis.

See figure 3 for a satellite in an elliptical orbit around the Earth with semi-major axis a and semi-minor axis b. The maximum velocity u_1 is the velocity at the closest approach to Earth, a point called the perigee, where $r = r_1$. For the farthest distance from Earth at the apogee, $r = r_2$ with the minimum orbital velocity u_2.

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[1] Acceleration is labeled with the subscript 'c' for centripetal since 'a' without a subscript is reserved to represent the semi-major axis in this paper.

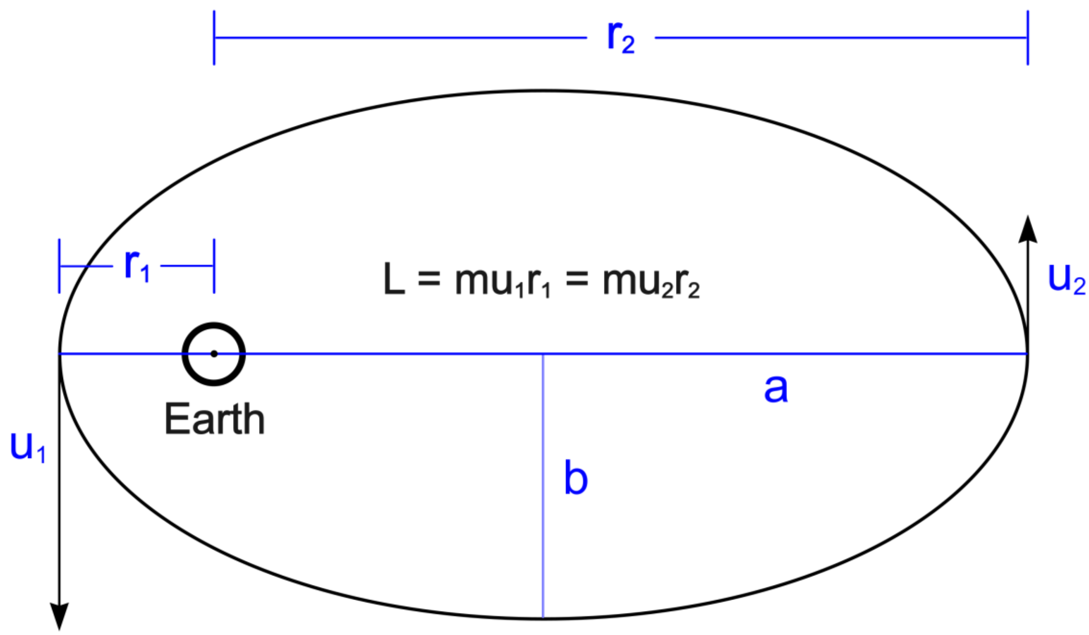

Figure 3. A satellite in an elliptical orbit around the Earth with semi-major axis a and semi-minor axis b. The angular momentum *L* takes on a simple form at the extreme orbital points since the velocity vector is perpendicular to the radial vector at each of the extreme points.

The authors have learned from Carl E Mungan and Murray S Korman of the United States Naval Academy [9] that many years ago Anthony French [10] included a homework problem in his mechanics text so that students can show with simple algebra that the generalization of equation (3) for the ellipse is

$$E = -\frac{GMm}{2a}, \tag{4}$$

where a is the semi-major axis. The trick is to recognize that at the extremes, perigee and apogee, also called the turning points, the velocity vector is perpendicular to the radial distance vector. Therefore, angular momentum is given by $L = mvr$ at the turning points and equation (1) becomes

$$E = \frac{L^2}{2mr^2} - \frac{GMm}{r}, \tag{5}$$

substituting $v = \frac{L}{mr}$. Note that equation (5) only applies at the turning points where $L = mu_1 r_1 = mu_2 r_2$. The next step outlined by French [10] is to write equation (5) at the perigee (r_1) and again for the apogee (r_2) to obtain two equations. Then the two simultaneous equations are solved to find E by eliminating L. However, Lawrence Evans at Duke University [11] showed the authors a very clever trick that further simplifies the algebra in a most elegant way. Write equation (5) as

$$r^2 + \frac{GMm}{E} r - \frac{L^2}{2mE} = 0 , \tag{6}$$

where the roots are the turning points. Therefore, $(r - r_1)(r - r_2) = r^2 - (r_1 + r_2) r + r_1 r_2 = 0$ is equivalent to equation (6). By comparing the r terms in each equation, $\frac{GMm}{E} = -(r_1 + r_2)$. Since $r_1 + r_2 = 2a$,

$$E = -\frac{GMm}{2a} , \tag{7}$$

the desired result obtained in essentially one line of algebra.

The spacecraft in figure 2 is initially in a parking circular orbit with velocity v_1 and radius r_1. The rocket engine must fire to increase the velocity by Δv_1 to achieve the speed necessary to shift the rocket to the elliptical orbit where the perigee and apogee are given by r_1 and r_2. See both figures 2 and 3. Comparing these figures, $u_1 = v_1 + \Delta v_1$ at the elliptical perigee distance $r = r_1$. The kinetic and potential energies for the elliptical orbit at the perigee can be set equal to the constant total energy given by equation (7):

$$\frac{1}{2}m(v_1+\Delta v)^2-\frac{GMm}{r_1}=-\frac{GMm}{2a}. \tag{8}$$

The velocity change Δv is the additional velocity needed for the satellite to enter the Hohmann transfer orbit. From equation (8),

$$(v_1+\Delta v)^2=\frac{2GM}{r_1}-\frac{GM}{a}. \tag{9}$$

Using the substitution $a=\frac{r_1+r_2}{2}$, after a few lines of algebra,

$$v_1+\Delta v=\sqrt{\frac{GM}{r_1}}\sqrt{\frac{2r_2}{r_1+r_2}}. \tag{10}$$

Employing $v_1=\sqrt{\frac{GM}{r_1}}$ according to equation (2) , equation (10) becomes

$$\Delta v_1=\sqrt{\frac{GM}{r_1}}\left[\sqrt{\frac{2r_2}{r_1+r_2}}-1\right]. \tag{11}$$

Equation 11 is a sophisticated result encountered usually in intermediate or advanced texts in mechanics. See the fine text *Orbital Motion* by Archie Roy [12], the first edition originally published in 1978. The year before this first edition was published Roy wrote an introductory calculus-based paper for teachers in this journal [13], where he discussed rocket dynamics, elliptical orbits in polar coordinates, and Hohmann transfer. He also derived equation (11) and the complementary equation for the velocity boost Δv_2 at the apogee in order to enter the higher circular orbit from the elliptical orbit (refer to figure 2).

In the next section, the theoretical velocity boost Δv_1 as calculated from equation (11) will be compared to the observational telemetry data found in the SpaceX videos for many flights [14]. The focus is on the velocity boost Δv_1 at the perigee where SpaceX supplies the telemetry data. The subsequent maneuvers to eventually reach the geostationary orbit take place much later. The students can calculate this time.

As an example, students can consider a satellite leaving a parking orbit with altitude $h_1 = 200\ \mathrm{km}$ above the Earth's surface to reach apogee on an elliptical journey to a geosynchronous height $h_2 = 36\ 000\ \mathrm{km}$ above the Earth's surface. The answer can be found with Kepler's Third Law $T^2 = \frac{4\pi^2}{GM}a^3$, where the result is $\frac{T}{2}$, the time for half the orbit. Using specialized units where the time is measured in 'months' (lunar periods) and distance in terms of the unit distance between the centers of the Earth and Moon, the calculation is faster, without needing values for G and M [15]. The answer is $\frac{T}{2} = 5\ \mathrm{h}$, rounded off to the nearest hour.

As mentioned earlier for the various SpaceX flights, the maximum altitudes above the Earth's surface for the destination orbits can be much less than $h_2 = 36\,000\ \mathrm{km}$ (GTO–), approximately $36\,000\ \mathrm{km}$ (GTO), or much greater (GTO+). These altitudes are reached long after the one-hour long video as found above with Kepler's Third Law. Rocket burns are done at that latter time to maneuver the satellite again to eventually circularise the orbit at the geosynchronous height of $36\,000\ \mathrm{km}$ above the Earth's surface.

Telemetry data and flight data

The values for the apogee distances of the Hohmann transfer orbit are given by SpaceX as distances measured from the Earth's surface [8], i.e. the altitude h_2. The value r_2 is found by adding the radius of the Earth R to the altitude: $r = R + h$. The radius of the Earth at the equator $(R = 6378\ \mathrm{km})$ is used in each case. The values for h_2 are listed in table 1 for many SpaceX flights. Final percentage errors are reported to two significant figures.

The observational values for h_1, v_1, and Δv_1 are obtained from the telemetry data in the appropriate SpaceX video [14]. See figure 4(a) for a screenshot from the Bangabandhu Mission and figure 4(b) for sample live telemetry data that appears in a section of the SpaceX videos. For the instant in figure 4(b), telemetry data indicates that the time is 1 min 10 s after launch and the rocket is traveling at $1503\ \mathrm{km \cdot h^{-1}}$ at an altitude of 11.8 km above the Earth's surface.

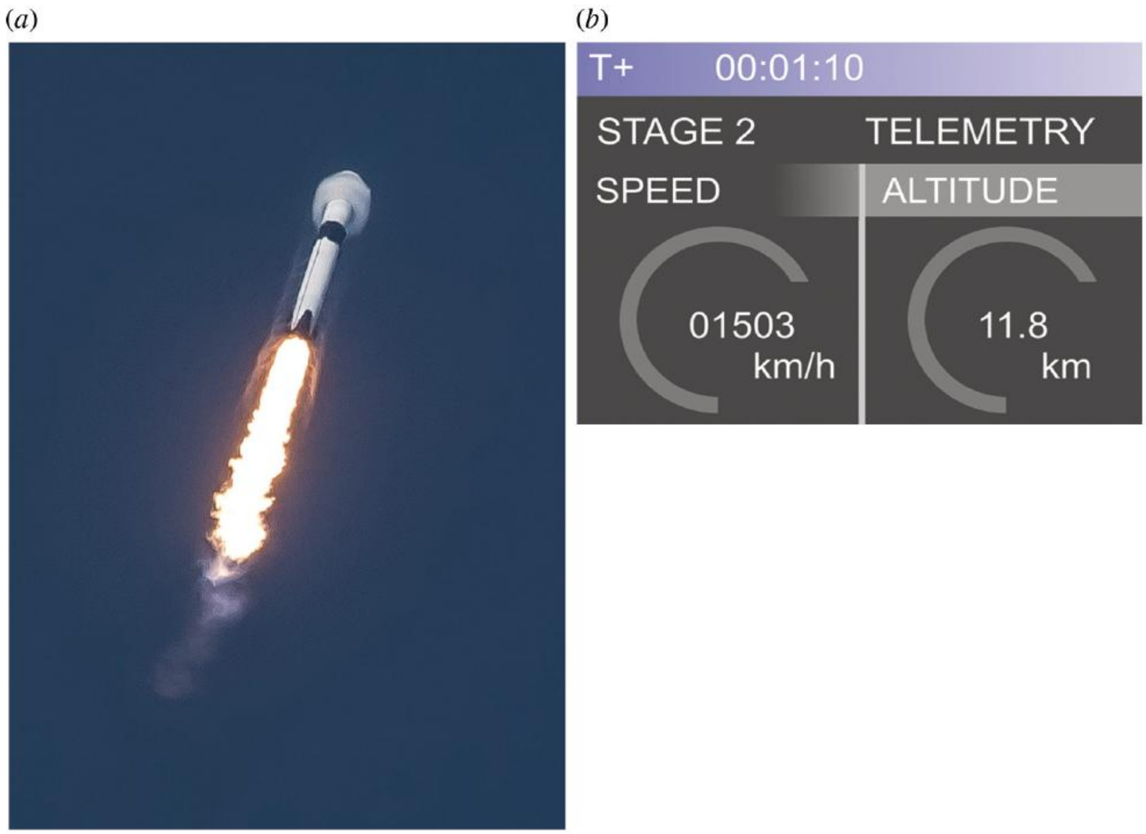

Figure 4. (a) Photo of the Falcon 9 carrying the Bangabandhu satellite into space, 11 May 2018. Reproduced with permission from [5]. (b) Sample telemetry data that appears in a section of the SpaceX videos. The telemetry data changes in real time, matching the speed and altitude of the spacecraft.

When the second stage cuts off for the first time, the rocket enters its parking circular orbit. SpaceX refers to this second stage engine cutoff as SECO-1, where the 1 indicates the first time the second stage engine has been turned off. However, the parking orbit is not precisely a circle as students will observe the velocity and height changing slightly over time. A typical time for the rocket to stay in the parking orbit is about 20 min. Therefore, the values for h_1 and v_1 are recorded after roughly 20 min and just before the engine fires up to give the spacecraft the necessary velocity boost to enter the elliptical transfer orbit. SpaceX refers to the second stage engine restart as SES-2. The first time the second stage started, SES-1, was during the initial flight to the parking orbit when it took over after the first stage engine.

Students will have fun finding h_1 and v_1 from the video telemetry as they will have to be alert to pause the video at the right instant. It took the authors several times to pause at the right place. The values for h_1 and v_1 entered in table 1 are therefore recorded just before SES-2, at which time the speed starts to rapidly change in the video telemetry section. The second stage will stay on for approximately a minute or less, depending on the specific flight. The student notes the final value for v_1 when the second stage cuts off for the second time, SECO-2. The values recorded in table 1 are the initial and final values for v_1 and the pair h_1 and h_2. The values for v_1 initial, v_1 final, Δv_1 obs, and h_1 are from the video telemetry, h_2 is from the flight launch reports [8], and Δv_1 th is calculated from equation (11) using h_1 and h_2.

Table 1. Velocity boosts to enter Hohmann transfer orbit: observation compared to theory. Columns v_1 initial, v_1 final, Δv_1 obs, and h_1 are from the video telemetry, h_2 is from the flight launch reports [8], and Δv_1 th is calculated from equation (11) using h_1 and h_2.

Description	v_1 initial (km/h)	v_1 final (km/h)	Δv_1 obs (m/s)	h_1 (km)	h_2 (km)	Δv_1 th (m/s)	PCT Error
F9-22 SES 9	26 430	35 254	2450	291	40 600	2500	2.0
F9-24 JCSAT 14	26 570	35 276	2420	198	35 957	2460	1.7
F9-25 Thiacom 8	26 448	36 440	2780	274	90 226	2850	2.6
F9-28 JCSAT 16	26 568	35 238	2410	198	35 912	2460	2.1
F9-35 Inmarsat 5 F4	26 418	36 095	2690	295	69 839	2750	2.2
F9-37 BulgariaSat 1	26 477	36 134	2680	250	65 640	2740	2.2
F9-39 Intelsat 35e	26 503	35 477	2490	248	42 742	2540	2.0
F9-43 EchoStar 105/SES 11	26 429	35 262	2450	288	40 519	2500	2.0
F9-45 Koreasat 5A	26 421	35 565	2540	288	50 185	2610	2.8
F9-49 Govsat 1	26 489	35 978	2640	249	52 000	2640	0.0
F9-55 Bangabandhu 1	26 392	34 840	2350	295	35 549	2420	3.0
F9-57 SES 12	26 457	35 859	2610	288	58 370	2680	2.7
F9-61 Merah Putih	26 579	34 900	2310	195	29 503	2340	1.3
F9-62 Telstar 18V	26 490	33 432	1930	257	18 060	1970	2.1
F9-64 Es'hail 2	26 595	35 457	2460	182	37 866	2490	1.2
F9-70 Nusantara	26 550	36 423	2740	208	69 036	2770	1.1
FH-2 Arabsat 6A	26 572	36 700	2810	198	89 815	2870	2.1

The observed velocity boost $\Delta v_1 = v_1(\text{final}) - v_1(\text{initial})$ and appears in the Δv_1 obs column. Note that the units have been converted to $m \cdot s^{-1}$. The theoretic value for Δv_1 is found from equation (11), inserting $r_1 = R + h_1$ and $r_2 = R + h_2$, where R is taken to be $R = 6378\ \mathrm{km}$, the radius of the Earth at the equator. All values for $\Delta v_1\ \mathrm{obs}$ and $\Delta v_1\ \mathrm{th}$ are rounded off to three significant figures. The percentage error $100\%\left[\dfrac{\Delta v_1\ \mathrm{th} - \Delta v_1\ \mathrm{obs}}{\Delta v_1\ \mathrm{obs}}\right]$ is found to be 3% or less. Sources of error include the idealization in the theoretic model of a perfectly circular parking orbit and an instantaneous velocity boost. The actual velocity boost takes about a minute in several of the missions.

The Falcon Heavy

The Arabsat 6A mission in table 1 used a transfer orbit with a very large apogee distance of 90 000 km above the Earth's surface before the orbit was finally circularised at the geosynchronous altitude of 36 000 km above the Earth's surface. SpaceX employed the Falcon Heavy for this job to make it easier to reach the initial 90 000 km above the Earth's surface. See figure 5 for the launch of this mission. Note the extra rocket boosters on the Falcon Heavy when compared the F9 in figure 1.

Figure 5. Falcon Heavy carrying the Arabsat 6A satellite into space. Reproduced with permission from [5].

A surprising feature of SpaceX videos is that booster rockets are recovered with dramatic return landings to the ground. In the Falcon Heavy video two side boosters are seen returning with multiple camera views. See figure 6 for a photo of the two boosters landing. Students can be given a history lesson of the Space Shuttle where boosters were recovered after falling into the ocean. It is amazing that SpaceX boosters land back on the ground like a science fiction movie. Be sure to watch this portion of the video showing multiple views of the two side boosters returning to Earth.

Figure 6. Two rocket boosters of the Falcon Heavy landing after use. Reproduced with permission from [5].

Conclusion

A real-life rocket application has been presented with associated theoretical analysis and telemetry flight data. Even students in pre-university physics classes can understand the mathematics, which only uses algebra. The starting point is the total energy for a satellite in a circular orbit, already included in the curriculum of an introductory physics course. A derivation is shown to generalize the energy equation for an elliptical orbit. Then, the formula giving the boost velocity necessary to place the rocket into a Hohmann transfer orbit is derived using simple algebra. The teacher can outline these derivations in order to let the students work out the algebra for themselves.

Students then watch exciting SpaceX videos to obtain observational data from telemetry in real time. The telemetric boost velocity is compared to the calculated value using the theoretical formula. Agreement is found to be 3% or less. Some error is expected since the theoretical model assumes an initial ideal circular orbit and an instantaneous velocity boost. The actual increase in velocity occurs over many seconds.

In summary, the activity outlined in this paper allows the student to connect physics to the engineering marvel of rocket science through theory and data. The assignment also includes the excitement of watching a rocket launch and rocket maneuvering in space.

Acknowledgments

The authors would like to thank photographer Ben Cooper for permission to use his excellent launch photographs [5].

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